STEINER MINIMAL TREES ON ZIG-ZAG LINES

BY

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ABSTRACT. A Steiner minimal tree for a given set P of points in the Euclidean plane is a shortest network interconnecting P whose vertex set may include some additional points. The construction of Steiner minimal trees has been proved to be an NP-complete problem for general P. However, the NP-completeness does not exclude the possibility that Steiner trees for sets of points with special structures can be efficiently determined. In this paper we determine the Steiner minimal trees for zig-zag lines with certain regularity properties. We also give an explicit formula for the length of such a tree.

1. Introduction. Let P denote a given set of points in the Euclidean plane. A *minimal spanning tree* for P is the shortest network (clearly, it has to be a tree) whose vertex set is P. A *Steiner minimal tree* (SMT) for P is the shortest network (again, it must be a tree) interconnecting P whose vertex set may include some additional points, called *Steiner points*, other than points in P (called *regular points*).

While there exist efficient algorithms for constructing minimal spanning trees, the construction of SMT's has been proved [3] to be an NP-complete problem. However, the NP-completeness does not exclude the possibility that the SMT's for sets of points with special structures can be efficiently determined. One such class was recently studied by Chung and Graham [2]. We quote:

> Perhaps the simplest infinite family of sets whose SMT's one might study are the ladders, so named by Boyce, who first focussed attention on them in [1]. A ladder L_n consists of $2n$ points arranged in a rectangular 2 by n array with adjacent pairs of points forming a square.

In their paper Chung and Graham determined the SMT's for L_n. However, even for sets of points with this extremely simple structure, it takes almost 30 pages [2] just to give a sketch of the proof. In this paper we study another class of sets of points for which SMT's can be determined with surprising ease.

A *zig-zag line* is a connected sequence of line segments $[p_1, p_2]$, $[p_2, p_3],\ldots,[p_{n-1}, p_n]$ which turn in alternate directions (see Figure 1). A zig-zag line is called *regular* if $\sphericalangle\ p_i p_{i+1} p_{i+2}$ is constant for all $i = 1, 2,\ldots,n - 2$. We will denote such a zig-zag line by $Z(P_n, \alpha)$ where $P_n = \{ p_1, p_2,\ldots,p_n\}$ and $\alpha < 180°$ is the constant angle. A zig-zag line is called *convex* if the points p_i are the vertices of a convex polygon. Note that for a convex zig-zag line, the line $[p_1, p_n]$ cuts through line $[p_i, p_{i+1}]$ for all i. A zig-zag line is called *normal* if $|p_i, p_{i+1}| \leqslant |p_i, p_{i+2}|$ for $1 \leqslant i \leqslant n - 3$ and $|p_{i+1}, p_{i+2}| \leqslant |p_i, p_{i+2}|$ for $2 \leqslant i \leqslant n - 2$, where $|p, q|$ denotes

the length of $[p, q]$. Note that if $Z(P_n, \alpha)$ is a minimal spanning tree for P_n, then $Z(P_n, \alpha)$ is normal (the converse is not true). Also note that being normal implies $\alpha \geqslant 60°$, and if $\alpha = 60°$, then $|p_2, p_3| = |p_3, p_4| = \cdots = |p_{n-2}, p_{n-1}|$.
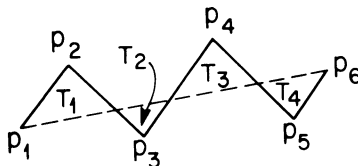

FIGURE 1. A convex zig-zag line

We now prove some properties of convex or normal $Z(P_n, \alpha)$.

LEMMA 1. *Let q be a point on $[p_i, p_{i+1}]$ of $Z(P_n, \alpha)$. Define $P' = \{p_1, \ldots, p_i, q\}$ and $P'' = \{q, p_{i+1}, \ldots, p_n\}$. If $Z(P_n, \alpha)$ is convex, so are $Z(P', \alpha)$ and $Z(P'', \alpha)$. If $Z(P_n, \alpha)$ is normal, so are $Z(P', \alpha)$ and $Z(P'', \alpha)$.*

PROOF. Immediate from the definitions.

It is interesting to note that if $Z(P_n, \alpha)$ is a minimal spanning tree for P_n, then $Z(P', \alpha)$ and $Z(P'', \langle)$ need not be minimal spanning trees for P' and P'', respectively. $\square$

LEMMA 2. *Suppose that $Z(P_n, \alpha)$ is normal. Then for $1 \leqslant i \leqslant n - 3$*

$$|p_i, p_{i+1}| \leqslant |p_i, p_j| \quad \text{for all } i + 2 \leqslant j,$$

where the equality is possible only when $j = i + 2$.

PROOF. $|p_i, p_{i+1}| \leqslant |p_i, p_{i+2}|$ by the definition of "normal". For $i + 3 \leqslant j$, let $[p_i, p_j]$ meet $[p_{i+2}, p_{i+3}]$ at q (see Figure 2).
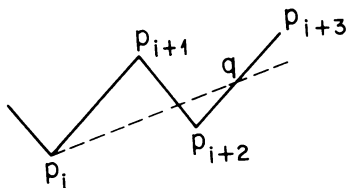

FIGURE 2. A normal zig-zag line

Then

$$\sphericalangle p_{i+1}qp_{i+2} \leqslant 180° - \alpha \leqslant 120°.$$

Therefore either $\sphericalangle p_{i+1}qp_i < 60° \leqslant \alpha < \sphericalangle qp_{i+1}p_i$, hence $|p_i, p_{i+1}| < |p_i, q| \leqslant |p_i, p_j|$, or $\sphericalangle p_{i+2}qp_i \leqslant 60° \leqslant \alpha < \sphericalangle qp_{i+1}p_i$, hence

$$|p_i, p_{i+1}| \leqslant |p_i, p_{i+2}| < |p_i, q| \leqslant |p_i, p_j|. \quad \square$$

Let $T_1, T_2, \ldots, T_{n-2}$ be the triangles formed by a convex $Z(P_n, \alpha)$ and $[p_1, p_n]$ (see Figure 1). Let S_i be the SMT for the three vertices of T_i and let $S = \bigcup_{i=1}^{n-2} S_i$. In this paper we prove that if $Z(P_n, \alpha)$ is convex and normal, then S is its SMT. We also give an explicit formula for the length of S.

2. Some preliminary results. Lemmas 3–7 are all well known in the literature [**4, 5**]. Let T denote an SMT.

LEMMA 3. *No two edges in T incident to a vertex can form an angle less than* $120°$.

LEMMA 4. *Each Steiner point in T has three edges meeting at* $120°$.

LEMMA 5. *T lies in the convex hull of P_n.*

LEMMA 6. *Let A, B and C be the three vertices of a triangle which has no angle exceeding $120°$. Then the length of an SMT for these three points is*

$$\left[|A, B|^2 + |B, C|^2 - 2 |A, B| \, |B, C| \cos(60° + \sphericalangle ABC) \right]^{1/2}.$$

LEMMA 7. *Suppose that T does not contain the edge $[p_i, p_j]$. Then $|p_i, p_j| \geqslant$ the length of any edge on the path joining p_i and p_j in the tree.*

We also quote a result from Pollak [**6**].

LEMMA 8. *Let four points A, B, C, D be given such that both possible full Steiner trees actually exist. Then the shorter of the two is in the same direction as the acute angle between the diagonals of the quadrilateral $ABCD$. The two full Steiner trees (given they both exist) will be equally long if and only if the diagonals are perpendicular.*

(A *full Steiner tree* with n given points is an SMT with $n - 2$ Steiner points.)

Let x_i denote $|p_{2i-1}, p_{2i}|$ and y_i denote $|p_{2i}, p_{2i+1}|$. We will suppress the subscript i when taking the sum over all i. Define $L_S(P_n, \alpha)$ to be the length of S.

THEOREM 1. *If $Z(P_n, \alpha)$ is convex and $60° \leqslant \alpha \leqslant 120°$, then*

$$L_S(P_n, \alpha) = \left[\left(\sum x \right)^2 + \left(\sum y \right)^2 - 2 \left(\sum x \right) \left(\sum y \right) \cos(60° + \alpha) \right]^{1/2}.$$

PROOF. Theorem 1 is reduced to Lemma 6 for $n = 3$. We prove the general case by induction on n. For $n > 3$, let $[p_1, p_n]$ meet $[p_2, p_3]$ at q. It is straightforward to verify $|p_2, q| = x_1 (\sum y)(\sum x)^{-1}$. Therefore

$$L_S(P_n, \alpha) = L_S(\{p_1, p_2, q\}, \alpha) + L_S(P_n - \{p_1, p_2\} \cup \{q\}, \alpha)$$

$$= \left[x_1^2 + x_1^2 \left(\sum y \right)^2 \left(\sum x \right)^{-2} - 2 x_1^2 \left(\sum y \right) \left(\sum x \right)^{-1} \cos(60° + \alpha) \right]^{1/2}$$

$$+ \left[\left(\sum x - x_1 \right)^2 + \left\{ \sum y - x_1 \left(\sum y \right) \left(\sum x \right)^{-1} \right\}^2 - 2 \left(\sum x - x_1 \right) \right.$$

$$\left. \cdot \left\{ \sum y - x_1 \left(\sum y \right) \left(\sum x \right)^{-1} \right\} \cos(60° + \alpha) \right]^{1/2}$$

$$= \left(x_1 + \sum x - x_1 \right) \left[1 + \left(\sum y \right)^2 \left(\sum x \right)^{-2} - 2 \left(\sum y \right) \left(\sum x \right)^{-1} \cos(60° + \alpha) \right]^{1/2}$$

$$= \left[\left(\sum x \right)^2 + \left(\sum y \right)^2 - 2 \left(\sum x \right) \left(\sum y \right) \cos(60° + \alpha) \right]^{1/2}. \quad \square$$

THEOREM 2. *Let $Z(P_n, \alpha)$ be convex, $60° \leqslant \alpha < 120°$. Suppose that q is a point on $[p_i, p_{i+1}]$ (q can be p_i). Then for $n - 2 \geqslant i \geqslant 2$,*

$$f(q) = L_S(\{p_1, p_2, \ldots, p_i, q\}, \alpha) + L_S(\{q, p_{i+1}, p_{i+2}, \ldots, p_n\}, \alpha)$$

is minimized when q is on $[p_1, p_n]$. Hence the minimum is equal to $L_S(P_n, \alpha)$.

PROOF. We first prove Theorem 2 for $i = 2$. Let $z = |p_2 q|$.

$$f(q) = \left[x_1^2 + z^2 - 2x_1 z \cos(60° + \alpha)\right]^{1/2}$$
$$+ \left[\left(\sum x - x_1\right)^2 + \left(\sum y - z\right)^2 - 2\left(\sum x - x_1\right)\left(\sum y - z\right)\cos(60° + \alpha)\right]^{1/2}.$$

Setting

$$\frac{\partial f(q)}{\partial z} = \left[z - x_1 \cos(60° + \alpha)\right]\left[x_1^2 + z^2 - 2x_1 z \cos(60° + \alpha)\right]^{-1/2}$$
$$- \left[\left(\sum y - z\right) - \left(\sum x - x_1\right)\cos(60° + \alpha)\right]$$
$$\cdot \left[\left(\sum x - x_1\right)^2 + \left(\sum y - z\right)^2 - 2\left(\sum x - x_1\right)\left(\sum y - z\right)\cos(60° + \alpha)\right]^{-1/2}$$

to zero, we obtain

$$\left(\frac{z - x_1 \cos(60° + \alpha)}{\sum y - z - (\sum x - x_1)\cos(60° + \alpha)}\right)^2$$
$$= \frac{x_1^2 + z^2 - 2x_1 z \cos(60° + \alpha)}{(\sum x - x_1)^2 + (\sum y - z)^2 - 2(\sum x - x_1)(\sum y - z)\cos(60° + \alpha)},$$

which can be reduced to

$$x_1^2\left[\left(\sum y - z\right)^2 - 2\left(\sum x - x_1\right)\left(\sum y - z\right)\cos(60° + \alpha)\right]$$
$$= \left(\sum x - x_1\right)^2\left[z^2 - 2x_1 z \cos(60° + \alpha)\right].$$

Clearly, $z = x_1(\sum y)(\sum x)^{-1}$ is the unique nonnegative solution.

We now prove the general case by induction on $n + i$. For $n = 4$, i is necessarily 2 and the proof has been given. For $n > 4$ and $i > 2$, let $[p_1, q]$ meet $[p_{i-1}, p_i]$ at r. Suppose that q is not on $[p_1, p_n]$; then neither is r. Hence

$$L_S(P_n, \alpha) < L_S(\{p_1, p_2, \ldots, p_{i-1}, r\}, \alpha) + L_S(\{r, p_i, p_{i+1}, \ldots, p_n\}, \alpha)$$
$$< L_S(\{p_1, p_2, \ldots, p_{i-1}, r\}, \alpha) + L_S(\{r, p_i, q\}, \alpha)$$
$$+ L_S(\{q, p_{i+1}, p_{i+2}, \ldots, p_n\}, \alpha)$$
$$= L_S(\{p_1, p_2, \ldots, p_i, q\}, \alpha) + L_S(\{q, p_{i+1} p_{i+2}, \ldots, p_n\}, \alpha),$$

where both inequalities follow from the induction hypothesis. This shows that q is not the minimizing point unless q is on $[p_1, p_n]$.

3. The main result. Let l be an edge in an SMT T. Then the *clockwise path* of l is the path starting with l and turning clockwise whenever possible. Similarly, we can define the *counterclockwise path* of l. Notice that both types of paths can end only at regular points.

REMARKS. For $\alpha \geqslant 120°$, $S_i = [p_i, p_{i+1}] \cup [p_{i+1}, p_{i+2}]$ and $L_S(P_n, \alpha) = \Sigma x + \Sigma y$. It is easily verified that $f(q) = L_S(P_n, \alpha)$ for all q.

THEOREM 3. *Suppose that* $Z(P_n, \alpha)$ *is convex and normal. Then* S *is its SMT.*

PROOF. We prove Theorem 3 by induction on n. Suppose that T is an SMT for P_n. Since $\sphericalangle p_2 p_1 p_3 < 180° - \sphericalangle p_1 p_2 p_3 = 180° - \alpha \leqslant 120°$, by Lemmas 1 and 3, p_1 has only one edge. If this edge ends at a regular point p_i, then

$$L_T = |p_1, p_i| + L_S(P_n - \{p_1\}, \alpha)$$
$$\geqslant x_1 + L_S(P_n - \{p_1\}, \alpha) \quad \text{by Lemma 2}$$
$$= L_S(\{p_1, p_2, p_2\}, \alpha) + L_S(P_n - \{p_1\}, \alpha)$$
$$> L_S(P_n, \alpha)$$

by Theorem 2 and by the remarks, hence either S is an SMT or T is not. Therefore we assume that p_1 is adjacent to a Steiner point s_1. Furthermore, the counterclockwise path of $[p_1, s_1]$ must end at p_2 for otherwise p_2 cannot be adjacent to any point. The reason for this is that p_2 certainly cannot be adjacent to any Steiner point s_2, for the clockwise path of $[p_2, s_2]$ would have nowhere to end. We may assume that p_2 is adjacent to p_3, for if p_2 is adjacent to any other regular point p_i, then we can replace $[p_2, p_i]$ by $[p_2, p_3]$ without increasing the length of T, by Lemma 2. But $[p_2, p_3]$ will cross the counterclockwise path of $[p_1, s_1]$, which violates the minimality of T.

Let H denote the counterclockwise path of $[p_1, s_1]$ which ends at p_2 and let s_2 be the Steiner point adjacent to p_2 ($s_1 = s_2$ is possible) on this path. We now show that we need only consider the case that H has only one Steiner point, i.e., $s_1 = s_2$.

We first prove that $\sphericalangle s_2 p_2 p_1 \leqslant 120°$. Suppose to the contrary that $\sphericalangle s_2 p_2 p_1 > 120°$. *Case* (i). $[p_2, s_2]$ meets $[p_3, p_4]$ at w. Then $\sphericalangle p_3 w p_2 = 180° - \sphericalangle w p_2 p_1 < 60° < \alpha$. Therefore we can replace $[p_2, s_2]$ by $[p_2, p_3]$ and shorten T, a contradiction to the minimality of T. *Case* (ii). $[p_2, s_2]$ ends before $[p_3, p_4]$ (see Figure 3).
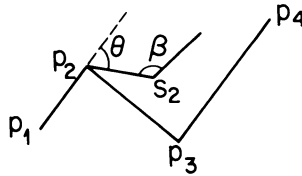

FIGURE 3. $\sphericalangle s_2 p_2 p_1 \geqslant 120°$

It must be the case that $\theta + \beta > 180°$, for otherwise the counterclockwise path of $[p_2, s_2]$ could not reach $[p_3, p_4]$ and would have nowhere to end. Furthermore $\beta = 120°$ by Lemma 2. Hence $\sphericalangle s_2 p_2 p_1 = 180° - \theta < 120°$.

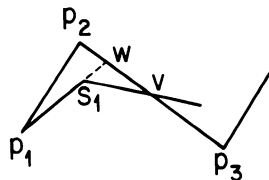Next we prove that $\sphericalangle p_2 p_1 s_1 < 60°$ (see Figure 4).

FIGURE 4. $\sphericalangle p_2 p_1 s_1 < 60°$

Suppose to the contrary that $\measuredangle p_2 p_1 s_1 \geq 60°$. We again consider two cases. *Case* (i). $[p_1, s_1]$ meets $[p_2, p_3]$ at w. Then

$$\measuredangle p_1 w p_2 = 180° - \alpha - \measuredangle p_2 p_1 w \leq 60° \leq \alpha.$$

Therefore $|p_1, w| \geq |p_1, p_2|$ and we can replace $|p_1, w|$ by $[p_1, p_2]$ and obtain a Steiner tree not longer than T. But we have already shown that p_1 cannot be adjacent to p_2 in an SMT, hence T is not minimal. *Case*(ii). $[p_1, s_1]$ ends before $[p_2, p_3]$. Let $[s_1, v]$ denote the edge at s_1 not on H. Then $[s_1, v]$ must meet $[p_2, p_3]$ or else the clockwise path of $[p_1, s_1]$ would have nowhere to end. Let the extension of $[p_1, s_1]$ meet $[p_2, p_3]$ at w. Then

$$120° = \measuredangle p_1 s_1 v > \measuredangle p_1 w p_3 = \alpha + \measuredangle p_2 p_1 s_1,$$

hence $\measuredangle p_2 p_1 s_1 < 60°$.

We have proven that $\measuredangle s_2 p_2 p_1 + \measuredangle p_2 p_1 s_1 < 120° + 60° = 180°$. Suppose that there are k Steiner points on H. Then since all angles of H are $120°$,

$$\measuredangle s_2 p_2 p_1 + \measuredangle p_2 p_1 s_1 = k \cdot (180° - 120°) = k \cdot 60°.$$

Hence k must be either one or two.

We first study the special case $n = 4$.

Claim. For $n = 4$, $k = 1$ in T, with the single exception that, if $\alpha = 60°$ and $x_1 = x_2 = y_1$, then there are two SMT's with $k = 1$ in one and $k = 2$ in the other.

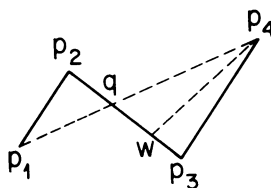PROOF. Let $[p_1, p_4]$ meet $[p_2, p_3]$ at q (see Figure 5).

FIGURE 5. A $Z(P_4, \alpha)$

Without loss of generality assume $|p_2, q| \leq |p_3, q|$. Let w be a point on $[p_3, q]$ such that $|p_2, q| = |w, q|$. Since $\measuredangle p_4 w p_2 \geq \alpha \geq \measuredangle p_4 p_2 w$ (the second inequality is due to the "normal" assumption), $|w, p_4| \leq |p_2, p_4|$. It follows easily from the cosine law that $\measuredangle p_4 q w \leq \measuredangle p_4 q p_2$. Furthermore, equality can take place only when $\alpha = 60°$ and $x_1 = x_2 = y_1$. By noting that the union of the SMT for $\{p_1, p_2, q\}$ and the SMT for $\{q, p_3, p_4\}$ is a full Steiner tree for $\{p_1, p_2, p_3, p_4\}$ with $k = 1$, the Claim follows immediately from Lemma 8. □

Next we show that for $n \geq 5$, $k = 1$ in T. Suppose to the contrary that $k = 2$ in T (see Figure 6).
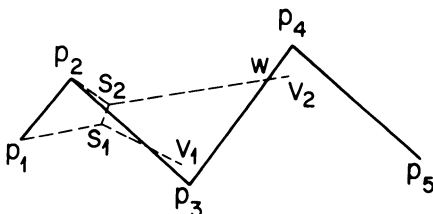

FIGURE 6. $k = 2$ and $n \geq 5$

We first prove that $[p_1, s_1]$ cannot meet $[p_3, p_4]$. Suppose to the contrary that they meet at c. Then $\sphericalangle p_1cp_3 = \sphericalangle p_2p_1c < 60° \leqslant \alpha < \sphericalangle cp_3p_1$. Therefore $|p_1, c| > |p_1, p_3| > |p_1, p_2|$, a contradiction to the minimality of T by Lemma 5.

Another fact we wish to point out is that the third edge at s_2, say $[s_2, v_2]$, must meet $[p_3, p_4]$. If not, let the extension of $[s_2, v_2]$ meet $[p_3, p_4]$ at w. Then

$$\sphericalangle s_2wp_4 = 180° - \sphericalangle s_2wp_3 = 180° - \sphericalangle p_2p_1s_1 > 120°.$$

Therefore the counterclockwise path of $[s_2, v_2]$ cannot reach p_4 and can end nowhere.

Let $[s_1, v_1]$ denote the third edge at s_1. Let $[s_2, v_2]$ meet $[p_3, p_4]$ at w. Suppose that $[s_1, v_1]$ meets $[p_3, p_4]$ at z. Let $[p_1, w]$ meet $[p_2, z]$ at q'. Then it is easily verified that $\sphericalangle p_1q'p_2 \leqslant \sphericalangle p_1qp_2$. By Lemma 8, the part of T connecting p_1, p_2, z, w can be shortened, a contradiction to the minimality of T, except possibly when $\alpha = 60°$, $x_1 = y_1 (= x_2)$, $z = p_3$, and $w = p_4$. Suppose that p_3 has a second edge in T. This edge must meet $[p_4, p_5]$, say at x, or else the counterclockwise path of this edge would have nowhere to end. But $\sphericalangle xp_3p_4 \leqslant \sphericalangle p_5p_3p_4 \leqslant \alpha$ implies that $|p_4, x| \leqslant |p_3, x|$, hence we can replace $[p_3, x]$ by $[p_4, x]$ without increasing the length of T. It follows that we can assume T is a combination of the tree $S_3 \cup S_4 \cup \cdots \cup S_{n-2}$ on $P'' = \{p_4, p_5, \ldots, p_n\}$ and an SMT on $P' = \{p_1, p_2, p_3, p_4\}$. Replacing the subtree of T connecting P' by the other full Steiner tree, i.e., $S_1 \cup S_2$, preserves the length of the tree according to the Claim. On the other hand Theorem 2 says that $L_S(P', 60°) + L_S(P'', 60°) > L_S(P, 60°)$. Hence T is not minimal.

Next consider the case that $[s_1, v_1]$ does not meet $[p_3, p_4]$. Let $[v_1, u]$ denote the edge at v_1 parallel to $[s_2, v_2]$. Note that the counterclockwise path of $[v_1, u]$ cannot end at p_4, since $[s_2, v_2]$ or its counterclockwise path ends there. Therefore $[v_1, u]$ must meet $[p_4, p_5]$ or else its counterclockwise path would have nowhere to end. Suppose that $[v_1, u]$ meets $[p_3, p_4]$ and $[p_4, p_5]$ at t and z, respectively. Since $\sphericalangle p_4tz = \sphericalangle p_2p_1s_1 < 60° \leqslant \alpha$, it follows that $|v_1u| > |t, z| > |p_4, z|$. Therefore we can delete $[v_1, u]$ and add $[p_4, z]$ to shorten T, a contradiction to the minimality of T.

We have shown that $k \neq 2$. Therefore $k = 1$, i.e., s_1 and s_2 are the same point s. Note that s must lie within $\triangle p_1p_2p_3$, for if s lies above $[p_2, p_3]$, then $\sphericalangle sp_2p_1 + p_2p_1s > \alpha \geqslant 60°$, a contradiction to the fact that $k = 1$. Let $[s, v]$ denote the third edge at s. Then $[s, v]$ must meet $[p_2, p_3]$ for otherwise the clockwise path of $[s, v]$ cannot reach p_3 and has no place to end. Next we prove that p_2 cannot have a second edge. Suppose to the contrary that $[p_2, w]$, $w \neq s$, exists. Then $[p_2, w]$ must meet $[p_3, p_4]$ at z, for otherwise the counterclockwise path of $[p_2, w]$ clearly would have nowhere to end (noting that $\sphericalangle p_1p_2w > \sphericalangle s_1p_2w \geqslant 120°$). Now $|p_2, p_4| \geqslant |p_3, p_4|$ implies $\alpha \geqslant \sphericalangle p_4p_2p_3 > \sphericalangle zp_2p_3$, hence $|p_2, z| > |p_3, z|$. Therefore we can replace $[p_2, z]$ by $[p_3, z]$ and shorten T, a contradiction to the minimality of T. We conclude that p_2 has only one edge.

Let $[s, v]$ meet $[p_2, p_3]$ at q. Then T is partitioned into a subtree connecting the three points p_1, p_2, q and a subtree connecting $q, p_3, \ldots, p_n$. By Theorem 2, q must lie on $[p_1, p_n]$ and the induction hypothesis brings the conclusion $T = S$. $\square$

COROLLARY. *The length of a Steiner minimal tree on* $Z(P_n, \alpha)$ *is*

$$\begin{cases} \left[\left(\sum x \right)^2 + \left(\sum y \right)^2 - 2 \left(\sum x \right) \left(\sum y \right) \cos(60° + \alpha) \right]^{1/2} & \text{if } 60° \leqslant \alpha \leqslant 120°, \\ \sum x + \sum y & \text{if } 120° \leqslant \alpha \leqslant 180°. \end{cases}$$

4. Some concluding remarks. Smith [7] studied $Z(P_n, \alpha)$ for the case $\alpha = 60°$ and all x_i and y_j are equal, say, of length one. He obtained SMT's for $n \leqslant 9$ and conjectured that for $n = 2k + 1$, the length of a Steiner minimal tree is $\sqrt{3} \, k$. Applying the formula given in the Corollary of Theorem 3 to this special case, we obtain the length of an SMT to be $\sqrt{3} \, k$ if $n = 2k + 1$, and $[3(k^2 + k) + 1]^{1/2}$ if $n = 2k + 2$. Therefore, Smith's conjecture is proved.

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